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Monotone Convergence of Operator Semigroups and the Dynamics of Infinite Particle Systems

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Convergence of strongly continuous contraction semigroups on a Banach space X is considered. The starting point is a family $\{A_\gamma\}_{\gamma\in\Gamma}$ of infinitesimal (semigroup) generators, indexed by a directed set Γ. If there is a dense subspace of vectors x such that $A_\gamma(x)$ is defined for γ sufficiently large, and $\lim_\gamma A_\gamma(x)$ exists, then additional conditions are considered which ensure the existence of an infinitesimal generator which may be regarded as the limit of the net $\{A_\gamma\}$. In case X is known to have a complete order structure, a monotone convergence theorem of a general nature is proved and it is shown how it applies to a particular existence problem for the dynamical semigroup in lattice gasses of classical statistical mechanics. A second type of results is also proved. These results are based on resolvent convergence and are applied to the corresponding existence problem in quantum statistical mechanics. Here the C^*-algebraic formalism is introduced, and the dynamics is given by a strongly continuous one-parameter group of *-automorphisms. At both levels (the classical and quantum), the solution to the time–Cauchy problem is obtained as a natural operator extension of the given partially defined, unbounded, infinitesimal operator. The extensions reflect particular boundary conditions for the problems under consideration. Finally, a distinguished, and canonical, extension operator is obtained, and its infinitesimal generator properties are analyzed. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let X be a Banach space, and let $A: \mathscr{D}(A) \to X$ be a linear operator which is defined on a dense domain $\mathscr{D}(A)$ in X. A well-known [16] and general result in the theory of operator semigroups states that the given operator A generates a strongly continuous semigroup of contraction operators $P(t, A): X \to X$ $(0 \leq t < \infty)$ iff A is *dissipative* and $\mathscr{R}(I - A) = X$. (For a linear operator B, $\mathscr{R}(B)$ denotes the range $\{Bx: x \in \mathscr{D}(B)\}$.) If X is

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also a Banach lattice (that is, equipped with a complete order lattice-structure [2, 25]), then there is analogous result [17] due to Phillips where an order theoretic notion takes the place of the dissipative condition. The relevant condition on A is "dispersive." It was first defined in [17], where the corresponding generation theorem is also proved: A linear operator $A: \mathscr{D}(A) \to X$ with dense domain in a Banach lattice X is the infinitesimal generator for a strongly continuous semigroup of positive contraction operators iff A is *dispersive* and $\mathscr{R}(I-A)=X$.

This theorem from [17] is true, in fact, only for Banach lattices slightly more special than the ones considered in what follows. Since we are concerned with approximation by positive semigroups, we need not go into the precise definition of dispersive operators: the generators of positive semigroups suffice for our present purpose; but it would be natural to consider approximations by the more general family of dispersive operators.

The existence problem of the dynamics, in terms of a Cauchy problem, can be solved successfully in applications [3–6, 11, 14, 18, 23] by the general generation results. The requirement on A to be dissipative, resp. dispersive, can frequently be read off quite easily: For the Kolmogorov differential equations, given by a matrix operator $A=(a_{ij})$ on an infinite discrete Banach lattice, the dispersive condition, for example, boils down to the inequalities $a_{ii} \leqslant 0$ and $a_{ij} \geqslant 0$ for $i \neq j$.

The other condition $\mathscr{R}(I-A)=X$ is almost always difficult to verify, and so for applications this is the real problem. Several successful approaches to the problem already exist [1, 3–5, 10, 16, 19, 23], and many of them are based on approximation. This is true in particular for the results in [6, 12, 17], but the limitation there is applicability only to discrete Banach lattices (i.e., $l^p(w)$-weighted sequence l^p-spaces), and this does not suffice for the existence problem in statistical mechanics. Our approach here is based on approximation, and we have results both for positive semigroups and for general contractive ones. Some more novel features in the present paper are the following:

(1) We consider extension operators $A \subset \tilde{A}$ which are infinitesimal generators and obtained through approximation. The quotient space $\mathscr{D}(\tilde{A})/\mathscr{D}(A)$ has the interpretation of physical boundary conditions.

(2) The extension operator $\tilde{A}$ is specified precisely in terms of the approximating operators. In [17] extension operators $\tilde{A}$ are obtained for Kolmogorov equations in $l^p(w)$-spaces, but there is no characterization of the extension $\tilde{A}$ whose existence only is proved. Classifications and explicit characterizations have not appeared even in special cases.

(3) Positivity properties are established here for a particular approximation (apparently new!) in the operator theoretic formulation of the dynamics of classical statistical mechanics (in particular lattices gasses).

The finite approximations which are commonly used [8, 9, 22] do not satisfy monotonicity conditions. Such monotonicity is verified in Section 8 for our particular and quite different approximation. As a consequence we obtain results of a more general nature.

(4) Our results apply to stochastic jump processes over any discrete set S of sites. In particular, our formulation applies equally well to any one of the cases $S = \mathbb{Z}^\nu$ for $\nu = 1, 2, \ldots$, appearing in models of lattice gases.

(5) Similarly, our conditions on the range of the considered particle interactions are of a general nature, and less restrictive than those previously analyzed.

(6) Our monotone convergence theorem applies to the general Banach (lattice) space setting, and is not restricted to semigroups in Hilbert space. Very few of the existing analogous Hilbert space results [7, 21] have been generalized to Banach space, in spite of the need which is imposed on the subject from applications [4, 10, 14, 19, 21, 23].

A price we pay for the additional generality is that, for the general models, we do not have uniqueness. However, the extension operators are specified precisely in terms of the approximating operators. In the applications mentioned under (4) and (5) above, the dynamical semigroup $P(t, \tilde{A})_{t \in [0,\infty)}$ acts, not in the sup-normed Banach (algebra) space $\mathscr{A} = C(\{0, 1\}^S)$, but rather in a complete Banach lattice X containing $\mathscr{A}$ as a dense subspace.

For infinite particle systems the time development is governed by a Markov process; however, the *existence* is non-trivial even for idealized models. In Section 8 we consider a stochastic model for lattice gasses: An infinite discrete set S is given, and the points in S correspond to sites for an infinite system of particles. Each site can be occupied by at most one particle. In time Δt a particle at site $x \in S$ attempts to jump with probability $c\,\Delta t + O(\Delta t)$. The speed c depends on the configuration, as well as on the transition probabilities $P(x, y)$, x, $y \in S$, defined in turn from the given interaction of the particles.

In this setting we obtain a strongly continuous, positive, contraction semigroup $P(t, A)$ as a solution to the dynamical differential equation. Properties of this semigroup are read off from the nature of the particular approximations employed, especially monotonicity.

2. Terminology

We shall denote by X a given complex Banach space. (The theory will work equally well for the real case.) A linear operator $A: \mathscr{D}(A) \to X$ with dense domain $\mathscr{D}(A)$ in X is said to be *dissipative* if one of the following three equivalent conditions is satisfied:

1. For all $x \in \mathscr{D}(A)$ there is *some* f in the dual X^* such that $f(x) = \|x\|$, $\|f\| = 1$ and

$$\operatorname{Re} f(Ax) \leqslant 0. \tag{2.1}$$

2. For all $x \in \mathscr{D}(A)$ and *all* $f \in X^*$, $\|f\| = 1$, such that $f(x) = \|x\|$ the inequality (2.1) holds.

3. The inequality

$$\|\lambda x - Ax\| \geqslant \lambda \|x\| \tag{2.2}$$

holds for all $\lambda \in \mathbb{R}_+$ and all $x \in \mathscr{D}(A)$.

The equivalence has been known since [1], and probably earlier; but it was not anticipated in the original paper on the subject [16]. The equivalence $1 \Leftrightarrow 2$ was used in a slightly different context in [11]. In the present paper we shall work almost entirely with the third formulation. It is clear from condition 3 that dissipative generators are closable and that the closure is again dissipative.

We define a *generalized Banach lattice* to be a Banach space X, together with a closed cone X_+, such that $\operatorname{span}(X_+) = X$. The order $x \leqslant y$ is defined by the requirement $y - x \in X_+$, and we impose further the following order-completeness assumption:

Every monotone increasing, and norm bounded, net $\{x_\gamma\} \subset X$ is convergent in X. (2.3)

(A *net* $\{x_\gamma\}$ is a set of elements, indexed by a directed set, Γ, say; and monotonicity is the requirement: $x_{\gamma_1} \leqslant x_{\gamma_2}$ for $\gamma_1 \leqslant \gamma_2$ in Γ.)

Our generalized Banach lattices are less restrictive than the ones considered by Phillips in [17]; and Phillips' lattices, in turn, are less restrictive than the ones defined in Birkhoff's book [2].

If X is obtained as the completion of a space of functions on some set K, then we say that X is a *function lattice* if the order on functions $f_1(k) \leqslant f_2(k)$, $k \in K$, implies the X-ordering, i.e., $f_2 - f_1 \in X_+$. The main result in [17] concerns the case of *discrete* K, while our attention will be directed towards the case of a *compact*, but non-discrete, space K. Typically K is the Tychonoff space $\{0, 1\}^S$ of functions defined on some infinite discrete set S, and taking values in the two point set $\{0, 1\}$ [26].

A mapping $t \to V(t)$ from $[0, \infty)$ into the algebra of bounded operators $\mathscr{B}(X)$ on X is said to be a *strongly continuous* (C_0, for short) *semigroup* if $V(0) = I$, $V(s+t) = V(s)\, V(t)$ for $s, t \in [0, \infty)$, and if $t \to V(t)$ is continuous into $\mathscr{B}(X)$ with respect to the strong operator topology [26]. Strong continuity will in the sequel be specified by the symbol C_0.

A C_0-semigroup is determined uniquely by the corresponding *infinitesimal generator*, A, say, where

$$Ax = \frac{d}{dt} V(t)\, x|_{t=0+}$$

for vectors x such that the right-derivative exists. This reflects the sense in which the semigroup generalizes the time–Cauchy problem from partial differential equations and probability.

It is well known [26] that A is a densely defined closed operator. Moreover, A is a dissipative operator iff the semigroup is contractive, i.e., $\|V(t)x\| \leqslant \|x\|$ for all $x \in X$, $t \in [0, \infty)$. We shall only consider contraction semigroups.

The semigroup generated by a given operator A is denoted by $V(t, A)$. If $\lambda \in \mathbb{C}$, and $\operatorname{Re} \lambda > 0$, then the operator $\lambda I - A$ has a bounded inverse which is denoted $R(\lambda, A) = (\lambda I - A)^{-1}$. It is given by the $\mathscr{B}(X)$-valued integral

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} V(t, A)\, dt. \tag{2.4}$$

3. A Preliminary Lemma

A given C_0-contraction semigroup $V(t, A)$ is said to be *positive* if the underlying Banach space X is a generalized Banach lattice, and if the cone X_+ is invariant under $V(t, A)$ for all $t \in [0, \infty)$. It is clear from (2.4) that $V(t, A)$-invariance of X_+ is equivalent to $R(\lambda, A)$-invariance.

The following lemma is therefore only a slight variant of a standard result in semigroup theory [13, p. 428; 26, p. 269].

Lemma 1 [6, Theorem 3.1]. *Let X be a Banach space, and let $\lambda \to R(\lambda)$ be a $\mathscr{B}(X)$-valued function defined on $\mathbb{R}_+$. Assume the following three conditions are satisfied*:

(a) $R(\lambda) - R(\mu) = (\mu - \lambda)\, R(\lambda)\, R(\mu)$, $\lambda, \mu \in \mathbb{R}_+$.

(b) $\lambda R(\lambda)$ *is contractive*, $\lambda \in \mathbb{R}_+$.

(c) $\lim_{\lambda \to \infty} \lambda R(\lambda) x = x$ *for all* $x \in X$.

Then there is an infinitesimal generator A such that $V(t, A)$ is C_0-contraction semigroup and $R(\lambda, A) = R(\lambda)$. If conversely A is known to be an infinitesimal generator, then $R(\lambda, A)$ satisfies conditions (a) *through* (c).

If X is also a generalized Banach lattice, then the added positivity requirement in (b), *i.e., $R(\lambda)\, X_+ \subset X_+$, is equivalent to positivity of the semigroup.*

In Section 7 we consider the case where X is a C^*-algebra and $R(\infty)$ satisfies the so-called Reynolds identity. We study the relevance of conditions (a) and (b) (in this setting) for the dynamical one-parameter groups of *-automorphisms.

4. Monotone Convergence

In many applications there is often a natural dense subspace where the infinitesimal equations for the dynamics are well defined (without convergence problems). In the Kolmogorov equations [6, 12, 17], the ∞ by ∞-matrix infinitesimal propagator $A=(a_{ij})$ is certainly defined on the space of finite vectors, i.e., sequences $f=(f(i))$ satisfying $f(i)=0$, except for a finite number of i's. For infinite partial systems [8, 9, 14, 22] the time process is always given by an infinitesimal propagator which can again be given on a dense domain $\mathscr{D}$ of nice functions, $\mathscr{D}=\{f: f$ defined on the compact space $\{0, 1\}^S$, and depending only on a finite number of coordinates$\}$.

Finally, in the C^*-algebraic framework [1, 3, 10, 11, 18–20], the corresponding operators A are typically the unbounded normal derivations.

The problem is to approximate A by a natural net of nice operators: To obtain extension operators $\tilde{A}$ for A such that $\tilde{A}$ is the infinitesimal generator of a C_0-contraction semigroup, and at the same time $\tilde{A}$ is the limit of the approximating net. Generally, the extensions are not unique [3, 4, 11, 23], and uniqueness requires relatively strong conditions on the interaction and on the other parameters in the problem.

Under monotonicity conditions we show that it is possible to choose a canonical, and natural, generator extension. In Section 4 we first show existence. The monotone extension is then characterized in Section 5. Finally, in Section 7, we apply the result to the infinite particle models, described in the introduction. We show that different approximations are possible for the model. Among the lattice gas approximations we locate a monotone net of operators.

THEOREM 2. *Let X be a generalized Banach lattice, and let $\{P(t, A_\gamma)\}_{\gamma\in\Gamma}$ be a net of positive, strongly continuous, contraction semigroups, indexed by a directed set Γ. Assume that for $\gamma_1 \leqslant \gamma_2$ in Γ the inclusion* $\mathscr{D}(A_{\gamma_1})\cap X_+ \subset \mathscr{D}(A_{\gamma_2})$ holds, and $A_{\gamma_1}(x) \leqslant A_{\gamma_2}(x)$ for $x\in\mathscr{D}(A_{\gamma_1})\cap X_+$.

Let A be a linear operator with dense domain $\mathscr{D}(A)$ in X. Assume further that $\mathscr{D}(A) \subset \bigcup_\gamma \mathscr{D}(A_\gamma)$, and that

$$A(x) = \lim_\gamma A_\gamma(x) \qquad \text{for all} \quad x \in \mathscr{D}(A). \tag{4.1}$$

Then there is an extension operator $\tilde{A}$, extending A, such that $\tilde{A}$ is the infinitesimal generator of a strongly continuous, positive, contraction semigroup $P(t, \tilde{A})$.

Moreover, it is possible to choose $\tilde{A}$ such that

$$\text{st.}\lim_\gamma P(t, A_\gamma) = P(t, \tilde{A}) \tag{4.2}$$

where this strong limit is monotone, *and uniform for t in compact subintervals of* $[0, \infty)$.

Proof. Since, for each γ, the operator A_γ is the infinitesimal generator of a C_0 positive contraction semigroup $P_t^\gamma = P(t, A_\gamma)$, there is a corresponding positive resolvent family $(\lambda > 0)$ $R(\lambda, A_\gamma) = (\lambda - A_\gamma)^{-1}$. Hence, for every $x \in X_+$ (the positive cone), the element $R(\lambda, A_\gamma)x$ falls in $X_+ \cap \mathscr{D}(A_\gamma)$. If $\gamma_1 \leqslant \gamma_2$, then the assumption in the theorem implies that the operator $D(\gamma_2, \gamma_1) = A_{\gamma_2} - A_{\gamma_1}$ is defined on $R(\lambda, A_{\gamma_1})x$, and $D(\gamma_2, \gamma_1)$ $R(\lambda, A_{\gamma_1})x \in X_+$. The second resolvent operator $R(\lambda, A_{\gamma_2})$ now again takes this element into X_+. We claim that

$$R(\lambda, A_{\gamma_2})x - R(\lambda, A_{\gamma_1})x = R(\lambda, A_{\gamma_2})\, D(\gamma_2, \gamma_1)\, R(\lambda, A_{\gamma_1})x. \tag{4.3}$$

Suppose, for the moment, that this identity (4.3) has been verified. Then it follows that the net $\{R(\lambda, \gamma)x\}_{\gamma \in \Gamma}$, for fixed λ, is increasing with values in X. Since $\|R(\lambda, A_\gamma)x\| \leqslant \lambda^{-1} \|x\|$ for all γ, the net is also bounded. It is then convergent by the assumed completeness of the lattice structure on X.

Hence, for fixed λ, the operator family $\{R(\lambda, A_\gamma)\}$ is strongly convergent. But the resolvent identity

$$R(\lambda, A_\gamma) - R(\mu, A_\gamma) = (\mu - \lambda)\, R(\lambda, A_\gamma)\, R(\mu, A_\gamma) \tag{4.4}$$

is satisfied for γ. If $R(\lambda) = \text{st.lim}_\gamma\, R(\lambda, A_\gamma)$, then the identity (4.4) becomes

$$R(\lambda) - R(\mu) = (\mu - \lambda)\, R(\lambda)\, R(\mu)$$

in the limit.

An easy argument shows that each operator $\lambda R(\lambda)$, for $\lambda > 0$, is a positive contraction. To apply Lemma 1, we need therefore only to check the convergence

$$\text{st.}\lim_{\lambda \to \infty} \lambda R(\lambda) = I \qquad \text{(the identity operator on } X\text{)}. \tag{4.5}$$

In the verification of (4.5), recall that each $\lambda R(\lambda)$ is contractive: By a Three-Epsilon Argument it suffices, in fact, to check the limit property on a dense of vectors, and we check it for x in the dense domain $\mathscr{D}(A)$.

For $x \in \mathscr{D}(A)$, consider the identity:

$$\lambda R(\lambda, A_\gamma)(\lambda - A_\gamma)x = \lambda x.$$

Using that each $\lambda R(\lambda, A_\gamma)$ is contractive, and by (4.1) that $\lim_\gamma A_\gamma(x) = A(x)$, we may take the limit in γ, and deduce the identity

$$R(\lambda)(\lambda - A)x = x, \tag{4.6}$$

or equivalently

$$\lambda R(\lambda)x = x + R(\lambda)\, A(x).$$

Here the second term tends to zero as $\lambda \to \infty$, since $\|R(\lambda)\, A(x)\| \leqslant \lambda^{-1}\, \|A(x)\|$, and the desired formula

$$\lim_{\lambda \to \infty} \lambda R(\lambda)x = x$$

results. A final 3ε argument (described above) then completes the proof, and shows that the norm-limit is indeed valid for all $x \in X$.

The lemma applies and it follows that, for some operator $\tilde{A}$, the formula

$$R(\lambda) = (\lambda - \tilde{A})^{-1} = R(\lambda, \tilde{A})$$

holds. Moreover, $\tilde{A}$ is the infinitesimal generator of a positive C_0 contraction semigroup $P(t, \tilde{A}) = e^{t\tilde{A}}$ $(0 \leqslant t < \infty)$, where

$$R(\lambda) = \int_0^\infty e^{-\lambda t} P(t, \tilde{A})\, dt.$$

To see that A extends the original operator A, we go back to identity (4.6). First note that $\mathscr{D}(A)$ is contained in $\mathscr{R}(R(\lambda)) = \mathscr{D}(\tilde{A})$. Hence for $x \in \mathscr{D}(A)$ we may apply $\lambda - \tilde{A} = R(\lambda)^{-1}$ to both sides of (4.6), and the desired conclusion

$$(\lambda - A)x = (\lambda - \tilde{A})\, R(\lambda)(\lambda - A)x = (\lambda - \tilde{A})x$$

follows.

By the Trotter–Kato Theorem [26, Theorem IX.12.1, p. 269] we conclude semigroup convergence from resolvent convergence. Hence, the statement (4.2) of the theorem holds true. The convergence in (4.2) is in fact monotone. For $\gamma_1 \leqslant \gamma_2$, $t > 0$, $n = 1, 2, \ldots$, we have

$$\left(\frac{n}{t} R\left(\frac{t}{n}, A_{\gamma_1}\right)\right)^n \leqslant \left(\frac{n}{t} R\left(\frac{t}{n}, A_{\gamma_2}\right)\right)^n.$$

In the limit $n \to \infty$, the operator inequality

$$P(t, A_{\gamma_1}) \leqslant P(t, A_{\gamma_2})$$

results, and this is the required monotonicity.

For completion of the proof of the theorem only the verification of identity (4.3) remains: The starting point here is an element x in the cone X_+. The element $y = R(\lambda, A_{\gamma_1})x$ is in $\mathscr{D}(A_{\gamma_1}) \cap X_+$ by the positivity of the operator $R(\lambda, A_{\gamma_1})$. But $\mathscr{D}(A_{\gamma_1}) \cap X_+ \subset \mathscr{D}(A_{\gamma_2})$ and $A_{\gamma_1}(y) \leqslant A_{\gamma_2}(y)$. In other words, $D(\gamma_2, \gamma_1)$ $R(\lambda, A_{\gamma_1})x \in X_+$. In fact, the operator $D(\gamma_2, \gamma_1)$ $R(\lambda, A\gamma_1)$ is bounded by the Closed Graph Theorem.

However, the identity (4.3) is clearly equivalent to

$$x - (\lambda - A_{\gamma_2})\, R(\lambda, A_{\gamma_1})x = D(\gamma_2, \gamma_1)\, R(\lambda, A_{\gamma_1})x.$$

Substitution of $x = (\lambda - A_{\gamma_1})z$, $z \in \mathscr{D}(A_{\gamma_1})$, yields

$$(\lambda - A_{\gamma_1})z - (\lambda - A_{\gamma_2})z = D(\lambda_2, \gamma_1)z.$$

This completes the proof of (4.3).

5. Convergence of Resolvent Operators

We are concerned with a linear operator $A: \mathscr{D}(A) \to X$ defined on a dense domain in a Banach space X, and a corresponding net $\{A_\gamma\}_{\gamma \in \Gamma}$ of approximating operators. We require that the domains $\{\mathscr{D}(A_\gamma)\}$ form a directed family of spaces, and that every x in $\mathscr{D}(A)$ belongs to $\mathscr{D}(A_\gamma)$ for γ sufficiently large. (This is made precise below!) If it is known that

$$Ax = \lim_\gamma A_\gamma(x) \qquad \text{for all} \quad x \in \mathscr{D}(A), \tag{5.1}$$

then there is an associated natural maximal extension A_G of A, the *graph extension.*

In general A_G is merely a *relation*, but various regularity assumptions on the approximating family (e.g., that each A_γ is a dissipative operator!) imply that A_G is in fact an operator.

We say that a vector x is in $\mathscr{D}(A_G)$ if there is a vector y in X and a set $\{x_\gamma\}$, $x_\gamma \in \mathscr{D}(A_\gamma)$, such that $x_\gamma \to x$ and $\lim_\gamma A_\gamma(x_\gamma) = y$. Then by definition

$A_G x = y$. The extension A_G is of special interest because it inherits properties of the approximating family: dissipative, dispersive, derivation, and generator properties. Instead of (5.1) we shall work in this section with the weaker requirement only that $\mathscr{D}(A_G)$ is dense in X.

PROPOSITION 3. *Let $\{A_\gamma\}_{\gamma \in \Gamma}$ be a family of dissipative operators, indexed by a directed set Γ.*

Assume that the domain $\mathscr{D}(A_G)$ of the graph limit A_G is dense. (*Recall that, a priori, A_G is only known to be a relation*!) *Then it follows that A_G is a linear, dissipative, and closed* operator.

Remark 4. We shall consider in Theorem 7 the situation where each of the operators A_γ is a generator. (This is stronger than dissipative, of course!) Under mild restrictions the conclusion in Proposition 3 is also valid for A_G (in particular, A_G is an *operator*), but the domain $\mathscr{D}(A_G)$ in general is *not* dense.

Proof. By definition, a vector x in X is in the domain of A_G iff for some $y \in X$ it is possible to find a net $\{x_\gamma\}_{\gamma \in \Gamma}$ satisfying the following three conditions: $x_\gamma \in \mathscr{D}(A_\gamma)$, $x_\gamma \to x$, and $A_\gamma x_\gamma \to y$. Then $A_G(x) = y$. To show first that A_G is indeed a linear operator we note that linearity is clear, and assume for some $y \in X$ that the vector $(0, y) \in X \times X$ is in the graph of A_G. We then have $x_\gamma \in \mathscr{D}(A_\gamma)$, $x_\gamma \to 0$, and $A_\gamma(x_\gamma) \to y$. For $u \in \mathscr{D}(A_G)$, we take $u_\gamma \in \mathscr{D}(A_\gamma)$ such that $\lim_\gamma(u_\gamma, A_\gamma u_\gamma) = (u, A_G u)$ in $X \times X$, and $\lambda > 0$. Then

$$\|(I - \lambda^{-1} A_\gamma)(u_\gamma + \lambda x_\gamma)\| \geqslant \|u_\gamma + \lambda x_\gamma\|.$$

Taking $\gamma \to \infty$ gives $\|u - \lambda^{-1} A_G(u) - y\| \geqslant \|u\|$. Then, in the $\lambda \to \infty$ limit, we have $\|u - y\| \geqslant \|u\|$. This implies that $y = 0$, since it is possible to approximate y in norm by vectors u in the dense domain $\mathscr{D}(A_G)$.

To see that A_G is also dissipative, consider the norm-approximation $(x_\gamma, A_\gamma(x_\gamma)) \to (x, A_G(x))$ in $X \times X$. Then for $\lambda > 0$, $\|(I - \lambda^{-1} A_\gamma)(x_\gamma)\| \geqslant \|x_\gamma\|$. In the limit $\gamma \to \infty$, the estimate $\|(I - \lambda^{-1} A_G)(x)\| \geqslant \|x\|$ results; and this is one of the defining conditions for dissipativeness.

Finally, we note that it is quite clear from the definition that A_G is a closed operator. Note that this is true even though the approximating operators A_γ are not themselves closed.

COROLLARY 5. *Let $\{A_\gamma\}_{\gamma \in \Gamma}$ be a family of dissipative operators indexed by a directed set Γ. For A_G to be the infinitesimal generator of a C_0-contraction semigroup it is necessary and sufficient that the conditions below hold*:

(a) *$\mathscr{D}(A_G)$ is dense in X, and*

(b) $(\lambda - A_G)\,\mathscr{D}(A_G) = X$.

Proof. The conditions are necessary by semigroup theory. Assume that (a) and (b) are satisfied. Then A_G is a closed, dissipative *operator* with dense domain. Hence A_G is a generator by Phillips' theorem (cf. the Introduction).

We consider in the next result a convergence type where condition (b) above is automatic if the individual operators A_γ in the net are known to be generators.

LEMMA 6. *Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a net of operators in a Banach space X. Assume that each operator in $\{A_\gamma\}$ is the infinitesimal generator of a C_0-contraction semigroup in X, and that the strong operator-limit*

$$\operatorname{st.lim}_{\gamma} R(\lambda_0, A_\gamma) = R(\lambda_0) \tag{5.2}$$

exists for some $\lambda_0 > 0$.

(i) *Then the same limit $R(\lambda)$ exists for all $\lambda > 0$ (in fact for all λ in $\mathbb{C}_+ = \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda > 0\}$), the convergence being uniform for λ in compact subsets of $\mathbb{R}_+$ (resp. $\mathbb{C}_+$).*

(ii) *Moreover, the operator $(\lambda - A_G)^{-1}$ is everywhere defined on $X = \mathscr{R}(\lambda I - A_G)$, and*

$$(\lambda - A_G)^{-1} = R(\lambda) \qquad \text{for} \quad \lambda \in \mathbb{C}_+. \tag{5.3}$$

(*Note, however, that A_G itself is generally not an operator.*)

Proof. (i) Suppose that $R(\lambda_0) = \operatorname{st.lim}_\gamma R(\lambda_0, A_\gamma)$ exists as a bounded operator in X for some $\lambda_0 > 0$. We show that the following set $F = \{\lambda > 0$: $R(\lambda) = \operatorname{st.lim}_\gamma R(\lambda, A_\gamma)$ exists$\}$ is open and closed in $\mathbb{R}_+$. Since F is assumed to be non-empty, the identity $F = \mathbb{R}_+$ would follow, and the desired limit relation is established.

To show openness, suppose $\lambda_0 \in F$, and $|\lambda - \lambda_\gamma| < \lambda_0^{-1}$, $\lambda \in \mathbb{R}_+$. We have

$$R(\lambda, A_\gamma) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A_\gamma)^{n+1}, \tag{5.4}$$

and each term in this infinite series is strongly convergent as $\gamma \to \infty$, with limit $(\lambda_0 - \lambda)^n R(\lambda_0)^{n+1}$.

But we have the following uniform estimate: $\|(\lambda_0 - \lambda)^n R(\lambda_0)^{n+1}\| = |\lambda_0 - \lambda|\ \lambda_0^{-(n+1)}$. An easy 2ε argument, or a well-known Dominated Convergence Theorem for vector valued functons, now implies that the left-hand side of (5.4) is in fact strongly convergent.

To show that F is closed in $\mathbb{R}_+$, consider a convergent point sequence

$\lambda_n \to \lambda_0$, $\lambda_n \in F$, $\lambda_0 > 0$. Then, for $x \in X$, and n fixed, the net $\{R(\lambda_n, A)x\}_{\gamma \in \Gamma}$ is convergent, and a fortiori Cauchy, in X.

For $\varepsilon > 0$, choose n (large) such that $|\lambda_n - \lambda_0| \, (\lambda_0 - \lambda_n)^{-1} \, \|x\| < \varepsilon/3$. For this value of n, choose $\gamma_0 \in \Gamma$ such that $\|R(\lambda_n, A_{\gamma_2})x - R(\lambda_n, A_{\gamma_1})x\| < \varepsilon/3$ for all $\gamma_1, \gamma_2 \geqslant \gamma_0$. Then

$$\begin{aligned} &\|R(\lambda_0, A_{\gamma_2})x - R(\lambda_0, A_{\gamma_1})x\| \\ &\quad \leqslant \|R(\lambda_0, A_{\gamma_2})x - R(\lambda_n, A_{\gamma_2})x\| + \|R(\lambda_n, A_{\gamma_2})x - R(\lambda_n, A_{\gamma_1})x\| \\ &\qquad + \|R(\lambda_n, A_{\gamma_1})x - R(\lambda_0, A_{\gamma_1})x\| \\ &\quad \leqslant 2\, |\lambda_0 - \lambda_n| \, (\lambda_0 \lambda_n)^{-1} \, \|x\| + \varepsilon/3 < \varepsilon. \end{aligned}$$

For $\gamma = \gamma_1$, resp. γ_2, we used the resolvent formula (4.4) in estimating the corresponding two terms: $\|R(\lambda_0, A_\gamma)x - R(\lambda_n, A_\gamma)x\|$.

If λ is restricted to a compact subset K of $\mathbb{R}_+$, we may cover K by disks such that the series (5.4) is dominated. Hence, for $x \in X$, the limit $R(\lambda, A_\gamma)x \to R(\lambda)x$ is uniform in K.

(ii) The dissipative property of A_G (cf. Proposition 3) ensures that $(\lambda - A_G)^{-1}$ is indeed an operator. Its domain $\mathscr{R}(\lambda - A_G)$ is closed. Let $\lambda \in \mathbb{C}_+$, and $x \in X$, be given. We define $z_\gamma = R(\lambda, A_\gamma)x$ for $\gamma \in \Gamma$, and note that $z_\gamma \to R(\lambda)x = z$ by part (i) of the Lemma. But $A_\gamma z_\gamma = (A_\gamma - \lambda)\, z_\gamma + \lambda z_\gamma \to -x + \lambda z$. Hence $z \in \mathscr{D}(A_G)$ and $(\lambda - A_G)z = x$. We also have $(\lambda - A_G)^{-1} x = z = \lim_\gamma z_\gamma = \lim_\gamma R(\lambda, A_\gamma)(\lambda - A_\gamma)\, z_\gamma = R(\lambda)(\lambda - A_G)z = R(\lambda)x$, concluding the proof of the Lemma.

THEOREM 7. *Let $\{A_\gamma\}$ be a net of operators in a Banach space X, and suppose that each A_γ is the infinitesimal generator of a C_0-contraction semigroup $P(t, A_\gamma)$ in X.*

We assume existence of the strong limit st.lim $R(\lambda_0, A_\gamma)$ *for some $\lambda_0 > 0$, and hence for all $\lambda > 0$.*

Then $\mathscr{D}(A_G)$ is equal to the range of $R(\lambda_0) = \text{st.lim}_\gamma R(\lambda_0, A_\gamma)$, *and the following four conditions are equivalent*:

(a) *$\mathscr{R}(R(\lambda))$ is dense for some (and hence for all) $\lambda > 0$.*

(b) *$\mathscr{D}(A_G)$ is dense.*

(c) $\text{st.lim}_{\gamma \to \infty} \lambda R(\lambda) = I$.

(d) *A_G is the infinitesimal generator of a C_0-contraction semigroup $P(t, A_G)$, and*

$$P(t, A_G) = \operatorname*{st.lim}_{\gamma} P(t, A_\gamma), \tag{5.5}$$

where the semigroup convergence is uniform for t in compact subintervals of $[0, \infty)$.

Proof. Assume first that $\text{st.lim}_\gamma R(\lambda, A_\gamma) = R(\lambda)$ exists for some $\lambda > 0$. We show that

$$\mathscr{R}(R(\lambda)) = \mathscr{D}(A_G). \tag{5.6}$$

(The equivalence of (a) and (b) is immediate from this.)

One of the inclusions ($\subset$) in the identity (5.6) was already established in the proof of Lemma 6(ii). There we also showed that

$$(\lambda - A_G)\, R(\lambda) = I \qquad \text{(the identity operator on } X.) \tag{5.7}$$

For the other inclusion ($\supset$) in (5.6), consider a pair $(x, y) \in X \times X$ such that there is a net $\{x_\gamma)$ with $x_\gamma \in \mathscr{D}(A_\gamma)$ and $(x_\gamma, A_\gamma x_\gamma) \to (x, y)$. Then

$$R(\lambda, A_\gamma)(\lambda - A_\gamma)\, x_\gamma = x_\gamma \qquad \text{for all} \quad \gamma \in \Gamma. \tag{5.8}$$

Taking the limit in γ, and using a simple 2ε argument, we get

$$R(\lambda)(\lambda - A_G)x = x \tag{5.9}$$

and it is clear from this that $x \in \mathscr{R}(R(\lambda))$.

(a) $\Rightarrow$ (c) Suppose $\mathscr{R}(R(\lambda_0))$ is dense. For $\lambda > \lambda_0$ we calculate

$$\begin{aligned} \lambda R(\lambda)\, R(\lambda_0)z &= \text{st.lim}_\gamma\, \lambda R(\lambda, A_\gamma)\, R(\lambda_0, A_\gamma)z \\ &= \text{st.lim}_\gamma\, \lambda(\lambda_0 - \lambda)^{-1}\, (R(\lambda, A_\gamma)z - R(\lambda_0, A_\gamma)z) \\ &= \lambda(\lambda_0 - \lambda)^{-1}\, (R(\lambda)z - R(\lambda_0)z) \\ &\to R(\lambda_0)z \qquad \text{for} \quad \lambda \to \infty. \end{aligned}$$

In the limit $\lambda \to \infty$ (last step) we used that $\|(\lambda_0 - \lambda)^{-1}\, R(\lambda)z\| = \mathcal{O}(\lambda^{-1})$ at ∞. The operator family $\{\lambda R(\lambda)\}_{\lambda > 0}$ is uniformly norm bounded by 1, and convergent, $\lambda R(\lambda) \to I$, on the dense subspace $\mathscr{R}(R(\lambda_0))$, and hence on all of X.

(c) $\Rightarrow$ (a) Let $f \in X^*$ and suppose that $f(R(\lambda_0)z) = 0$ for all $z \in X$. Then

$$\begin{aligned} f(R(\lambda)z) &= f(R(\lambda_0)z + (\lambda_0 - \lambda)\, R(\lambda_0)\, R(\lambda)z) \\ &= f(R(\lambda_0)[z + (\lambda_0 - \lambda)\, R(\lambda)z]) = 0 \end{aligned}$$

for all $\lambda > 0$. But $f(z) = \lim_{\lambda \to \infty} f(\lambda R(\lambda)z)$ by (c). Hence $f = 0$, and $\mathscr{R}(R(\lambda_0))$ is dense.

(a) $\Rightarrow$ (d) Assuming (a), we note that all the conditions in the Trotter–Kato Theorem are satisfied (cf. [26, p. 269, Theorem 1]). We conclude that there is some operator $\tilde{A}$ satisfying $R(\lambda)=(\lambda-\tilde{A})^{-1}$ for all $\lambda>0$. Moreover $\tilde{A}$ is the infinitesimal generator of a C_0-contraction semigroup $P(t, \tilde{A})$, and

$$P(t, \tilde{A})=\operatorname{st.lim}_{\gamma} P(t, A_\gamma),$$

the convergence being uniform for t in compacts $\subset [0, \infty)$.

But $\mathscr{D}(\tilde{A})=\mathscr{R}(R(\lambda))=\mathscr{D}(A_G)$, and $A_G x=\tilde{A}x$. Hence $\tilde{A}\subset A_G$. In fact, the two operators are equal by (a). Alternatively, we have a generator $\tilde{A}$ contained in a dissipative A_G (Proposition 3). But then equality $\tilde{A}=A_G$ follows from maximality of generators.

Remark 8. The *weak graph limit* A_W of an operator net A_γ is defined analogously to A_G, the only difference being the requirement of weak convergence instead of norm convergence: A pair $(x, y)\in X\times X$ is in the graph of A_W iff $\exists x_\gamma\in\mathscr{D}(A_\gamma)$ such that $\|x_\gamma-x\|\to 0$, and $A_\gamma x_\gamma\to y$ weakly. Then $A_W x=y$.

COROLLARY 9. *Suppose that the net* $\{A_\gamma\}$ *considered in Theorem* 7 *is in fact a sequence. Then*

$$\mathscr{R}(R(\lambda))=\mathscr{D}(A_W). \tag{5.10}$$

Hence, if

$$\mathscr{D}(A_W) \text{ is dense,} \tag{b_W}$$

then $A_W=A_G$ *and the properties* (*a*) *through* (d) *are satisfied.*

Proof. The problem is the inclusion $\mathscr{D}(A_W)\subset\mathscr{R}(R(\lambda))$. Hence, for $x\in\mathscr{D}(A_W)$ we consider $x_\gamma\in\mathscr{D}(A_\gamma)$ s.t. $\|x_\gamma-x\|\to 0$, and $A_\gamma x_\gamma\to A_W x$ weakly. Using $\langle f, R(\lambda, A_\gamma)(\lambda-A_\gamma)\, x_\gamma\rangle=\langle f, x_\gamma\rangle$ for $f\in X^*$, we show that $R(\lambda)(\lambda-A_W)x=x$. A 3ε argument is involved, and in taking the limit we require boundedness of $(\|A_\gamma x_\gamma\|+\|x_\gamma\|)$. This would follow from the Uniform Boundedness Principle if weak boundedness is known. Hence, the requirement that the net be a sequence.

Remark 10. Any of the conditions (a) through (c), or for sequences (b_W), imply that $R(\lambda)$ has trivial null-space $\mathscr{N}(R(\lambda))=\{x: R(\lambda)x=0\}=\{0\}$. But the condition

$$\mathscr{N}(R(\lambda))=\{0\} \tag{n}$$

alone does not imply any of the four equivalent properties (a) through (d). However, an easy argument establishes the following equivalence:

$$\text{Condition (n) is satisfied iff } A_G \text{ is an operator.} \tag{5.11}$$

Even though A_G is in general only a relation, its inverse $(\lambda - A_G)^{-1} = R(\lambda)$ is always an operator. For the proof of (5.11), suppose $y \in X$ and $(0, y)$ belongs to the graph of A_G. By (5.9) we then have $R(\lambda)(\lambda 0 - y) = 0$, and therefore $y = 0$, assuming (n). Similarly, if A_G is known to be an operator, condition (n) is trivially satisfied by (5.7). In fact, we have proved the equivalence

$$(0, y) \in \operatorname{graph}(A_G) \Leftrightarrow R(\lambda) y = 0 \qquad \text{for all} \quad \lambda > 0. \tag{5.11'}$$

THEOREM 11. *Let $\{A_\gamma\}$ be a net of generators, i.e., each A_γ is the infinitesimal generator for a C_0-contraction semigroup in a Banach space X. Suppose that the strong limit, $R(\lambda) = \operatorname{st.lim}_\gamma R(\lambda, A_\gamma)$ exists for some λ, and hence all, $\lambda > 0$. Let H denote the closure of the range $R(\lambda)$. (This is λ-independent.)*

(i) *Then*

$$H = \{x \in X: \lim_{\lambda \to \infty} \|\lambda R(\lambda) x - x\| = 0\}. \tag{5.12}$$

(ii) *There is a strongly continuous contraction semigroup $P(t, \tilde{A})$ in H with generator $\tilde{A}$ (a densely defined operator in H) such that*

$$R_H(\lambda, \tilde{A}) = (\lambda - \tilde{A})^{-1} = R(\lambda) \mid_H$$

(restriction to H).

(iii) *We have*

$$\mathscr{D}(\tilde{A}) = \{x \in \mathscr{D}(A_G): A_G x \in H\} \tag{5.13}$$

and

$$\tilde{A} x = A_G x \qquad \textit{for} \quad x \in \mathscr{D}(\tilde{A}), \tag{5.13'}$$

where A_G is the graph limit of the original net $\{A_\gamma\}$.

PROBLEM 12. Does the $\mathscr{B}(X)$-strong limit $\operatorname{st.lim}_{\lambda \to \infty} \lambda R(\lambda)$ exist?

Proof. (i) One of the inclusions $(\subset)$ in (5.12) has already been established. Indeed we showed convergence for vectors x in the dense $\mathscr{R}(R(\lambda_0))$, and the general case follows by an easy 3ε argument. The other inclusion $(\supset)$ follows from the observation that $\mathscr{R}(R(\lambda))$ is λ-independent.

(ii) We begin by the existence problems for the semigroup $P(t, \tilde{A})$. Since the operators $R(\lambda)$ satisfy the resolvent identity (or rather pseudo-resolvent), we note in particular that $R(\lambda)$ and $R(\lambda')$ commute for different values of λ and λ'. By the definition of H (H = the closure of $\mathscr{R}(R(\lambda_0))$ for a particular $R(\lambda_0)$ it follows that H is invariant under $R(\lambda)$ for all $\lambda \in \mathbb{C}_+$. If we set $R_1(\lambda) = R(\lambda)|_H$ (restricted to H), then the operator family $\{R_1(\lambda)\} \subset \mathscr{B}(H)$ is a pseudo-resolvent in H. But the implication, (a)$\Leftrightarrow$(c), of Theorem 7 amounts to the conclusion

$$\operatorname{st.}\lim_{\lambda \to \infty} \lambda R_1(\lambda) = I_H \qquad \text{(the identity operator in } H\text{).} \tag{5.14}$$

When Theorem 7 is applied to $R_1(\lambda)$, it follows that $R_1(\lambda) = (\lambda - \tilde{A})^{-1} = R_H(\lambda, \tilde{A})$ for some infinitesimal generator $\tilde{A}$ in H. That is, there is a C_0-contraction semigroup $P(t, \tilde{A})$ in H with corresponding infinitesimal generator $\tilde{A}$, and resolvent family $R_1(\lambda)$. (See also Lemma 1.)

(iii) Recall that, in general, A_G is only a relation in X. Indeed, if $\mathscr{G}(A_G)$ denotes the graph of A_G, we established in Remark 10 the equivalence (5.11′)

$$(0, y) \in \mathscr{G}(A_G) \Leftrightarrow y \in \mathscr{N}(R(\lambda)).$$

Consider the relation B obtained from A_G by restriction to H. More precisely, the graph of B is defined by

$$\mathscr{G}(B) := \mathscr{G}(A_G) \cap (H \times H). \tag{5.15}$$

We now make the following three claims:

$$B \text{ is an operator;} \tag{5.16}$$

i.e., there are no vectors in $\mathscr{G}(B)$ of the form $(0, y)$ with $y \neq 0$.

$$\tilde{A} \subset B; \tag{5.17}$$

i.e., the graph of the generator $\tilde{A}$ obtained in (i) is contained in $\mathscr{G}(B)$.

$$B \text{ is dissipative as an operator in } H. \tag{5.18}$$

Assume for the moment that each of the three claims have been proved. Then the equality $\tilde{A} = B$ follows, since the generator $\tilde{A}$ is maximal dissipative. This equality in turn is clearly equivalent to the conclusions (5.13) and (5.13′) stated in part (ii) of the Theorem.

Let $y \in H$ and assume $(0, y) \in \mathscr{G}(B)$. Then, a fortiori, $(0, y) \in \mathscr{G}(A_G)$, and by (5.11′) in Remark 10 this is equivalent to $R(\lambda)y = 0$ for all $\lambda > 0$. But by (i), $\lambda R(\lambda)y \to y$ as $\lambda \to \infty$, and hence $y = 0$. This proves (5.16).

In proving (5.17) we establish first the weaker fact: $\mathscr{G}(\tilde{A}) \subset \mathscr{G}(A_G)$. This would suffice by (5.15), since $\mathscr{G}(\tilde{A})$ is contained in $H \times H$ by definition. For $x \in \mathscr{D}(\tilde{A})$ we have $R(\lambda)(\lambda - \tilde{A})x = R_1(\lambda)(\lambda - A)x = x = R(\lambda)(\lambda - A_G)x$. Here we used (5.9) and the following chain of inclusions: $\mathscr{D}(\tilde{A}) = \mathscr{R}(R_1(\lambda)) \subset$

$\mathscr{R}(R(\lambda))=\mathscr{D}(A_G)$. When $R(\lambda)\,\lambda x$ is cancelled above, the identity $R(\lambda)\,\tilde{A}x=R(\lambda)\,A_G x$ follows. Hence

$$\tilde{A}x=\lim_{\lambda\to\infty}\lambda R(\lambda)\,A_G x. \tag{5.19}$$

We also have $\lambda R(\lambda)\,A_G x=A_G\lambda R(\lambda)x$, and $\lambda R(\lambda)x\to x$ as $\lambda\to\infty$. But A_G is closed, and in view of (5.19), therefore $\tilde{A}x=A_G x$. This proves (5.17), and in particular (5.13)–(5.13′).

Using (5.17), the dissipative property of B is clear. Indeed, for $x\in\mathscr{D}(B)$ we have $\|\lambda x-Bx\|=\|\lambda x-A_G x\|\geqslant\lambda\,\|x\|$.

6. Applications

In this section we apply the two Banach space results to particular problems in approximation. First we show that the extension $\tilde{A}$ obtained in Theorem 2 in fact coincides with the graph extension A_G. We then consider briefly an application to the theory of unbounded derivations in C^*-algebras [11, 18, 19]. In particular, a result of Sakai [20] on commutative derivations is extended in two ways.

COROLLARY 13. *Let $P(t,\tilde{A})$ be the positive C_0-contraction semigroup with infinitesimal generator $\tilde{A}$, obtained in Theorem 2. Then $\tilde{A}$ is equal to the graph extension A_G.*

Proof. In the course of the proof of Theorem 2 we showed that the strong limit $\text{st.lim}_\gamma\, R(\lambda, A_\gamma)$ exists for $\lambda>0$. Hence the conclusion of the corollary $\tilde{A}=A_G$ follows immediately from Theorem 7.

In non-commutative C^*-algebras the convergence condition (5.2) in Theorem 7 is easy to verify for commutative derivations [20]. Hence Sakai's result of [20] is a corollary, but Theorem 7 gives additional information on the generator.

A *normal derivation* in a $UHF-C^*$-algebra $\mathscr{A}$ is a linear transformation $\delta\colon\mathscr{D}(\delta)\to\mathscr{A}$ with dense domain $\mathscr{D}(\delta)$ in the C^*-algebra $\mathscr{A}$. The defining axioms are:

$\mathscr{D}(\delta)$ is a self-adjoint subalgebra of $\mathscr{A}$ containing the identity, (6.1)

$$\delta(a^*)=\delta(a)^*, \tag{6.2}$$

$$\delta(ab)=\delta(a)b+a\delta(b) \qquad \text{for} \quad a,b\in\mathscr{D}(\delta). \tag{6.3}$$

If there is an increasing sequence $(\mathscr{A}_n)$, $n=1,2,\ldots$, of finite type I factors

$\mathscr{A}_n$ with $\mathscr{D}(\delta) = \bigcup \mathscr{A}_n$, then we say that δ is *normal.* It follows (cf. [18]) that for each n, there is an element $h_n = h_n^* \in \mathscr{A}$ such that

$$\delta(x) = [ih_n, x] = ih_n x - ixh_n \qquad \text{for all} \quad x \in \mathscr{A}. \tag{6.4}$$

Commutative derivations form a special class of normal derivations: If it is possible to choose the h_n's to be mutually commuting, we say that δ is *commutative.*

COROLLARY 14. *Let δ be a* normal *derivation in a UHF* $-$ *C^*-algebra $\mathscr{A}$, and let $\{h_n\}$, $n = 1, 2, \ldots$, be an associated defining sequence of hermitean elements, chosen according to* (6.4).

Let $\delta_n = \mathrm{ad}(ih_n)$, and let δ_G be the graph limit of the sequence of operators $\{\delta_n\}$, $n = 1, 2, \ldots$.

*Assume that the strong limits $R_\pm = \mathrm{st.lim}_n (I \pm \delta_n)^{-1}$ exist as bounded operators on $\mathscr{A}$. Then δ_G is the infinitesimal generator of a strongly continuous one-parameter group of *-automorphisms $\{\alpha_t\}_{t \in \mathbb{R}}$ of $\mathscr{A}$, and*

$$\alpha_t(x) = \lim_n e^{ith_n} x e^{-ith_n} \tag{6.5}$$

for all $x \in \mathscr{A}$, and $t \in \mathbb{R}$. This limit is uniform for t in compact subintervals of $\mathbb{R}$.

Proof. By Theorem 7 we have semigroups $\beta_t(\pm)$ with generators $\zeta_\pm$ extending $\pm\delta$, respectively. Each of the semigroups is given by the limit of formula (6.5), and, as a consequence, they piece together to a C_0-one-parameter group of *-automorphisms. The infinitesimal generator is in fact equal to δ_G, again by Theorem 7.

The following result is of a more general nature, but it applies in particular to approximations with bounded derivations of the form $\delta_n = \mathrm{ad}(ih_n)$, $h_n = h_n^* \in \mathscr{A}$.

COROLLARY 15. *Let X be a C^*-algebra and let $\{A_\gamma\}$ be a net of bounded, or unbounded, *-derivations. Assume that each A_γ is the generator of a C_0-one-parameter group of *-automorphisms. Finally, assume that the strong limit*

$$\mathrm{st.lim}_\gamma R(\lambda, A_\gamma) = R(\lambda)$$

exists for at least two values $\lambda_\pm$ of λ: $\lambda_+ > 0$, and $\lambda_- < 0$.

Then conclusions (i) *through* (iii) *of Theorem* 11 *are valid with the following added information*:

(a) *The closed subspace H defined in* (i) *is in fact C^*-subalgebra.*

(b) *$P(t, \tilde{A})$ is a C_0-one-parameter group of *-automorphisms of H, where $\tilde{A}$ is the *-derivation constructed from A_G by formulas* (5.13) *and* (5.13′); *or in terms of graphs,*

$$\mathscr{G}(\tilde{A}) = \mathscr{G}(A_G) \cap (H \times H). \tag{6.6}$$

Proof. The proof is direct from Theorem 11 in view of the following observation: Let (x^1, y^1) and (x^2, y^2) be elements in $\mathscr{G}(A_G)$, the graph of the graph limit A_G. Then the element $(x^1x^2, x^1y^2 + y^1x^2)$ in $X \times X$ is also in $\mathscr{G}(A_G)$. Here x^1x^2 denotes the product in X of the elements x^1 and x^2. For the proof of the observation, consider nets $(x_\gamma^i)_{\gamma \in \Gamma}$, $i = 1, 2,...,$ of elements $x_\gamma^i \in \mathscr{D}(A_\gamma)$ and assume that $\lim_\gamma(x_\gamma^i, A(x_\gamma^i)) = (x^i, y^i)$ in $X \times X$. Then $(x^1x^2, x^1y^2 + y^1x^2) = \lim_\gamma(x_\gamma^1x_\gamma^2, A_\gamma(x_\gamma^1x_\gamma^2))$ in view of the derivation identity for the individual *-derivations A_γ.

Similarly one can easily show that (x^*, y^*) is in $\mathscr{G}(A_G)$ for all $(x, y) \in \mathscr{G}(A_G)$. If we knew that $\delta = A_G$ were also an operator, the it would follow that δ satisfies the derivation identities (6.2) and (6.3).

The formula, $H = \overline{\mathscr{D}(A_G)}$, shows that H is a C^*-subalgebra of X. But we know from (iii) in Theorem 11 that the restriction $\tilde{A}$, given by (6.6), is an operator. Since the algebraic properties for $\tilde{A}_G$ trivially carry over to the graph-restriction, it follows that $\tilde{A}$ is indeed a *-derivation (with dense domain in H). We also know by the argument above that $\tilde{A}$ generates a C_0-one-parameter group of bounded invertible operators in H. It follows by a simple calculus argument that this group $P(t, \tilde{A})$ is in fact a group of *-automorphisms. Indeed, for $x, y \in \mathscr{D}(\tilde{A})$, the *-derivation property of $\tilde{A}$ allows us to conclude:

$$\frac{d}{ds} P(t - s, \tilde{A})(P(s, \tilde{A})(x)\, P(s, \tilde{A})(y)) = 0.$$

Hence, for each $t \in \mathbb{R}$, the function under the d/ds-derivative is constant in s, and the *-automorphism property is immediate from this.

7. Reynolds' Identity

The interplay between the graph limits and the resolvent perturbations is useful in other areas of the theory of unbounded derivations. Positivity, and complete positivity, of the generalized resolvent operators (pseudo-resolvents) enter at different levels in these applicatons [11]. Here we shall isolate the pseudo-resolvents of Lemma 1 and Theorem 11.

Let $\mathscr{A}$ be a C^*-algebra with unit e, and let $P_e(\mathscr{A})$ be the positive unit preserving linear mappings of $\mathscr{A}$ into itself. (We have $T \in P_e(\mathscr{A})$ iff T is linear, $Te = e$, and $T\mathscr{A}_+ \subset \mathscr{A}_+$, where $\mathscr{A}_+ = \{x \in \mathscr{A}: x = x^*, \text{spec}(x) \geqslant 0\}$. Indeed, the listed conditions for T imply that T is contractive, $\|T\| \leqslant 1$, by a well-known theorem in C^*-algebras.)

DEFINITION 16. Let $\lambda \to Q(\lambda)$ be a $P_e(\mathscr{A})$-valued function define on the real line $\mathbb{R}$. Assume $Q(0) = I$, the identity operator $\mathscr{A} \to \mathscr{A}$. Define $R(\lambda) = \lambda^{-1} Q(\lambda)$ for $\lambda \neq 0$. We say that Q is a *Reynolds system* provided the following two conditions hold:

$$R(\lambda) - R(\mu) = (\mu - \lambda)\, R(\lambda)\, R(\mu). \tag{7.1}$$

The operator $R = R(1)$ satisfies the identity (7.2)

$$(Rx)(Ry) = R[x(Ry) + (Rx)y - (Rx)(Ry)]$$

for all elements x and y in $\mathscr{A}$. (Here the product is that of the algebras $\mathscr{A}$.)

Let Q be a Reynolds system, and define subspaces $\mathscr{A}_0 = \{x \in \mathscr{A}: R(\lambda)x = 0\}$ and $\mathscr{A}_1 = \{R(\lambda)x: x \in \mathscr{A}\}$ $(\lambda \neq 0)$. It is then clear that $\mathscr{A}_1$ is *-subalgebras of $\mathscr{A}$. Each subspace is λ-independent for $\lambda \neq 0$. If $\mathscr{B}$ denotes the norm-closure of $\mathscr{A}_1$, then $\mathscr{B}$ is $Q(\lambda)$-invariant C^*-algebra, and the restriction $Q_1(\lambda) = Q(\lambda)\,|_{\mathscr{B}}$ is a Reynolds system on $\mathscr{B}$. We say that $(Q_1, \mathscr{B})$ is the *restricted system.*

SCHOLIUM 17. *Let $(Q, \mathscr{A})$ be a Reynolds system of a C^*-algebra $\mathscr{A}$, and let $(Q_1, \mathscr{B})$ be the restricted system. Then*

(i) *There is a closed *-derivation δ satisfying $Q(\lambda) = (I - \lambda^{-1}\delta)^{-1}$ iff $\mathscr{A}_0 = \{0\}$.*

(ii) *For the restricted system we have $\mathscr{B}_0 = \{0\}$.*

(iii) *In the general case, we have the following identities:*

$$\mathscr{B} = \{x \in \mathscr{A}: \lim_{\lambda \to \infty} Q(\lambda)x = x\}$$
$$= \{x \in \mathscr{A}: \lim_{\lambda \to -\infty} Q(\lambda)x = x\}.$$

(iv) *The following relations hold:*

$$\mathscr{B} \cap \mathscr{A}_0 = \{0\} \tag{7.3}$$

and

$$\mathscr{B} \cdot \mathscr{A}_0 = \mathscr{A}_0 \cdot \mathscr{B} = \mathscr{A}_0 \tag{7.4}$$

where $\mathscr{B}\cdot\mathscr{A}_0=\{b\cdot a\colon b\in\mathscr{B},\ a\in\mathscr{A}_0\}$, *and the product* $b\cdot a$ *is that of the algebra* $\mathscr{A}$.

(v) *There is a* C_0*-one-parameter group of *-automorphisms* α_t *of* $\mathscr{B}$ *with infinitesimal generator* δ_1 *in* $\mathscr{B}$, $\sigma_t=e^{t\delta_1}$, *such that* $Q_1(\lambda)=(I-\lambda^{-1}\delta_1)^{-1}=\lambda\int_0^\infty e^{-\lambda t}\alpha_t\,dt$.

(vi) *In particular,* $Q(\lambda)=(I-\lambda^{-1}\delta)^{-1}$ *with a *-derivation generator* δ *in* $\mathscr{A}$ *iff* $\mathscr{B}=\mathscr{A}$.

(vii) *The identity*

$$R(\lambda)\,xR(\lambda)y=R(\lambda)[xR(\lambda)y+(R(\lambda)x)y-\lambda R(\lambda)\,xR(\lambda)y] \tag{7.5}$$

is valid for all $\lambda\in\mathbb{R}\backslash\{0\}$.

(viii) *If the strong limit*

$$\operatorname*{st.lim}_{\lambda\to\infty} Q(\lambda)=Q \tag{7.6}$$

exists, then Q *is a projection with range equal to* $\mathscr{B}$, *and satisfying the conditional* expectation-identity

$$Q(ba)=bQ(a)\qquad\text{for all}\quad b\in\mathscr{B},\ a\in\mathscr{A}. \tag{7.7}$$

Proof. The proofs of (i) through (v) involve only modifications of those which appear in Section 5, and they will therefore largely be omitted. However, we will comment on some of the modified proofs involving the so-called Reynolds identity (7.2).

For each λ we can define an operator $\delta(\lambda)=\lambda-R(\lambda)^{-1}$ (with domain equal to $\mathscr{A}_1$) iff $\mathscr{A}_0=0$. This is so because $\mathscr{A}_0=\mathscr{N}(R(\lambda))$ and $\mathscr{A}_1=\mathscr{R}(R(\lambda))$ for all λ. But it is immediate from the resolvent identity (7.1) that $\delta(\lambda)$ is independent of λ. Hence $\delta(\lambda)=\delta(1)=I-R^{-1}$. A simple calculation involving the Reynolds identity (7.2) now shows that $\delta=\delta(1)$ satisfies Leibnitz' formula:

$$\delta(ab)=\delta(a)b+a\delta(b)\qquad\text{for all } a,b\in\mathscr{A}_1. \tag{7.8}$$

Closedness of δ and the hermitean property [(6.1)–(6.2)] $\delta(a)^*=\delta(a)^*$ for $a\in\mathscr{A}_1$ follow easily.

Note that (ii) is in fact a consequence of (7.3) in (iv). As for (iii) we note that $\lim_{\lambda\to\pm\infty}Q(\lambda)x=x$ holds for $x\in\mathscr{A}_1$, and then for x in the norm-closure $\mathscr{B}$ by a 3ε argument. The other inclusion($\supset$) is trivial since the approximating elements $Q(\lambda)x$ belong $\mathscr{A}_1$.

Relation (7.3) follows from (iii).

Since $e\in\mathscr{B}$, it is enough in (7.4) to show that the product xz is in $\mathscr{A}_0$ for all $x\in\mathscr{A}_0$ and $z\in\mathscr{A}_1$. Consider $x\in\mathscr{A}_0$, and $z=Ry\in\mathscr{A}_1$. Then a substitution into (7.2) gives $0=R[xRy-0]=R(xz)$, which is the desired conclusion.

In view of (ii) we have the relation $Q_1(\lambda)=(I-\lambda^{-1}\delta_1)^{-1}$ for a closed *-derivation δ_1 in $\mathscr{B}$. By semigroup theory δ_1 is the generator of a C_0-one-parameter group of invertible operators $\mathscr{B}\to\mathscr{B}$. The *-derivation property of δ_1 implies that this is in fact a one-parameter group of *-automorphisms.

Conclusion (vi) is a special case of (v).

Identity (7.5) in (vii) follows from a substitution of (7.1), with $\mu=1$, into (7.2). Multiplying through in (7.5) by λ^2 and taking the limit $\lambda\to\infty$, assuming as in (viii) that the limit Q exists, we get the identity

$$(Qx)(Qy)=Q[x(Qy)+(Qx)y-(Qx)(Qy)]. \tag{7.9}$$

The projection property $Q^2=I$ follows from (iii). We may then derive (7.7), either from a polarization of (7.9), or else from a C^*-algebraic result of Tomiyama [25]. The Tomiyama theorem requires the norm identity $\|Q\|=1$, which in turn is immediate here.

Remark 18. Frequently the limit (7.6) will exist only in a weaker sense. As an example we mention the following result: Suppose a faithful invariant state φ exists, i.e., $\varphi(Rx)=\varphi(x)$ for $x\in\mathscr{A}$. Then R lifts to a contractive mapping, also denoted by R, in the Hilbert space $L^2(\mathscr{A},\varphi)$. The corresponding L^2-adjoint operator R^* can easily be shown to satisfy the identity $R^*+R=2R^*R$, which in turn implies that $U=I-2R$ is L^2-isometric. Let the limit projection $\lim_n(1/(n+1))\sum_{k=0}^n U^k$ (which exists by the mean ergodic theorem) be denoted by $I-\tilde{Q}$. Then it follows that $\tilde{Q}$ is a conditional expectation which projects $L^2(\mathscr{A},\varphi)$ onto $L^2(\mathscr{A}_1,\varphi)=L^2(\mathscr{B},\varphi)$.

8. Infinite Classical Particle Systems

Let S be an infinite discrete set (for example, the ν-dimensional integer lattice $\mathbb{Z}^\nu$), and let K denote the *compact* infinite Cartesian product space $\{0,1\}^S$ with product topology, where $\{0,1\}$ is the two-point set with elements 0 and 1. Let $C(K)$ denote the sup-normed Banach space of continuous functions on K. Let $F\subset C(K)$ be the subalgebra of functions in $C(K)$ which depend on a finite set of coordinates. More precisely, $f\in C(K)$ belongs to F iff there is a finite subset S', depending on f, in S, and a function f' in as many variables as there are elements in S' such that $f(\eta)=f'(\{\eta(s)\}_{s\in S'})$ for all $\eta\in K$. Clearly F is dense in $C(K)$ by the Stone–Weierstrass Theorem.

A generalized Banach lattice X containing F as a dense subspace is said to be a *function lattice* if the norm is weaker than that of $C(K)$, and $F_+\subset X_+$. In other words, the defining requirement on the order is that every f in F, satisfying $f(\eta)\geqslant 0$ for all $\eta\in K$, must be a positive element in X.

A lattice gas over S is given by a pair of functions P, c, where

(1) P is the transition function, defined on $S \times S$ and satisfying $P(x, y) \geqslant 0$, as well as $\sum_{y \in S} P(x, y) = 1$ for $x \in S$, and where

(2) c is a non-negative function defined on $S \times K$, continuous in second variable, which designates the speed; i.e., in time Δt a particle at x attempts to jump with probability $c\,\Delta t + O(\Delta t)$.

We say that the pair P, c is *admissible* if conditions (1) and (2) above are satisfied.

If X is associated to $F(S)$, then a $F-X$ *multiplier* is a function m defined on K with the following property:

$$\text{For all } f \in F \text{ the product } mf \text{ belongs to } X. \tag{8.1}$$

Here mf is the pointwise product defined by $(mf)(\eta) = m(\eta) f(\eta)$ for $\eta \in K$.

It is clear that a multiplier m defines a closable operator M in X with domain F, given by $Mf = mf$ for $f \in F$. If moreover $m(\eta) \leqslant 0$ for all $\eta \in K$, then $\bar{M}$ is the infinitesimal generator of a C_0-contraction semigroup $P(t, \bar{M})$ in X. Moreover the operator $P(t, \bar{M})$ for every $t \geqslant 0$ is given by the multiplier e^{tm}.

Let X be a function lattice over a system on S. Let P, respectively c, be the transition function, respectively the speed; i.e., a pair of functions satisfying properties (1) and (2) above.

For $\eta \in K$ we define $S_1(\eta) = \eta^{-1}(1) = \{x: \eta(x) = 1\}$, and $S_0(\eta) = \eta^{-1}(0)$. The set $\Omega(\eta) = S_1(\eta) \times S_0(\eta)$ then designates the possible transition pairs for a discrete model with sites that can contain at most one particle.

The process in X, given by P, c, is said to be *normal* if for all $\eta \in K$ the sum

$$\sum_{(x,y) \in \Omega(\eta)} c(x, \eta)\, P(x, y)$$

is convergent, and if

$$m(\eta) = \sum_{(x,y) \in \Omega(\eta)} c(x, \eta)\, P(x, y) \tag{8.2}$$

is a $F-X$ multiplier.

In order to define the infinitesimal generator of the process, we make use of the state $\eta_{x,y}$ obtained from a given state $\eta \in K$, and representing transition from site x to site y. More precisely,

$$\eta_{x,y}(z) = \begin{cases} \eta(z), & x \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y. \end{cases}$$

For $f \in F$ we then define

$$Af(\eta) = \sum_{(x,y) \in \Omega(\eta)} c(x, \eta)\, P(x, y)[f(\eta_{x,y}) - f(\eta)] \tag{8.3}$$

and assume that A is a dissipative operator when defined on the dense domain F in X.

Let Γ be the directed set of finite subsets of S. For a finite subset $\gamma \subset S$ we denote by χ_γ the indicator functon of $\gamma \times \gamma$; and $P_\gamma = \chi_\gamma P$ is then the function on $S \times S$ which is equal to P on $\gamma \times \gamma$, and vanishes on the complement.

Let T be a subset of S, and x, y a pair of points in S, $x \in T$. The set obtained from T by deleting the point x, and instead adding y, is denoted $T^{(x,y)}$.

Let M be the (diagonal-) operator, defined by the multiplier m given in (8.2), and let γ be a finite subset of S.

Make the following definitions:

$$a_\gamma(\eta, \varphi) = \begin{cases} c(x, \eta)\, P_\gamma(x, y) & \text{if } \eta \neq \varphi,\ (x, y) \in \Omega(\eta) \text{ and } S_1(\varphi) = S_1(\eta)^{(x,y)}, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_\gamma^0 f(\eta) = -Mf(\eta) + \sum_{\varphi \in K} a_\gamma(\eta, \varphi)\, f(\varphi). \tag{8.4}$$

Then each operator A_γ^0 is pre-closed with domain F, and the closure $A_\gamma = \overline{A_\gamma^0}$ generates a C_0-contraction semigroup.

THEOREM 19. *Let S be an infinite discrete set, and let X be a function lattice. Let P, c be an admissible pair, and assume that the corresponding process is normal. Finally, assume that the generator A, given by* (7.3) *above, is dissipative.*

Then the graph extension A_G of A, associated to the approximating family $\{A_\gamma\}$, is the infinitesimal genertor of a C_0-contraction semigroup of positive maps in X. Moreover,

$$P(t, A_G) = \operatorname{st.lim}_\gamma P(t, A_\gamma),$$

with uniform and monotone convergence for t in compact subintervals of $[0, \infty)$.

Proof. The proof is based on the previous discussion, and an application of Theorem 2 and Corollary 13.

Assumption (4.1) in Theorem 2 is known to be satisfied for the generator A, and the operator family $\{A_\gamma\}$, which were introduced in (8.3) and (8.4) above. This is simple to check directly, and we refer to [9] for a thorough discussion. (A short argument involves a particular approximating family of functions $\{c_\gamma\}_{\gamma \in \Gamma}$, $c_\gamma(x, \eta) \to c(x, \eta)$, such that $c_\gamma(x, \cdot)$ only depends on coordinates indexed by γ. If then M_γ is given by the multiplier $m_\gamma(\eta) = \sum_{(x,y) \in \Omega(\eta)} c_\gamma(x, \eta) P_\gamma(x, \eta)$, then we have a different approximation,

$$b_\gamma(\eta, \varphi) = \begin{cases} a_\gamma(\eta, \varphi) & \text{for} \quad \eta \neq \varphi, \\ -m(\eta) & \text{for} \quad \eta = \varphi, \end{cases}$$

and associated operators,

$$\begin{aligned} B_\gamma f(\eta) &= \sum_{\varphi \in K} b_\gamma(\eta, \varphi) f(\varphi) \\ &= \sum_{(x,y) \in \Omega(\eta)} c_\gamma(x, \eta) P_\gamma(x, \eta)[f(\eta_{x,y}) - f(\eta)] \to Af(\eta). \end{aligned}$$

But the normality assumption gives by a direct argument the approximation $(B_\gamma - A_\gamma)(f) \to 0$ for $f \in F$. The desired condition (4.1) now follows.

The approximation $B_\gamma \to A$ is the one which is common in the theory. Here we use instead the A_γ-approximation, and the reason is that the latter approximation is *monotone* in the sense:

$$A_{\gamma_1}(f) \leqslant A_{\gamma_2}(f) \qquad \text{for} \quad f \in F_+ \text{ and } \gamma_1 \leqslant \gamma_2. \tag{8.5}$$

This is the critical condition which is involved in the application of Theorem 2. In the verification of (8.5) we calculate, for $f \in F_+$ and finite subsets γ_1, γ_2 of S, $\gamma_1 \subseteq \gamma_2$, the difference of the two expressions (on either side), and use cancellation of diagonal terms:

$$\begin{aligned} A_{\gamma_2} f(\eta) - A_{\gamma_1} f(\eta) &= \sum_{\varphi \in K} [a_{\gamma_2}(\eta, \varphi) - a_{\gamma_1}(\eta, \varphi)] f(\varphi) \\ &= \sum_{\varphi} c(x, \eta)[P_{\gamma_2}(x, y) - P_{\gamma_1}(x, y)] f(\varphi) \\ &= \sum_{\substack{\varphi \\ (x,y) \in \gamma_2 \setminus \gamma_1}}{}' c(x, \eta) P(x, y) f(\varphi(x, y)) \geqslant 0, \end{aligned}$$

where the last primed sum is restricted to points $\varphi \neq \eta$ in K such that the corresponding transition pairs (x, y) fall in the relative complement, $\gamma_2 \setminus \gamma_1$, of γ_1 in γ_2. This completes the proof of (8.5), and Theorem 19 now follows directly from Theorem 2 and Corollary 13.

Note added in proof. Since the completion and acceptance of this paper (1980/1981), a number of related papers have appeared, some of which extend certain of our theorems. In particular, we wish to call attention to papers by O. Bratteli and D. W. Robinson, by C. J. K. Batty, by W. Arendt, P. Chernoff, and T. Kato, and, especially the recent paper, by L. L. Helms, Order properties of attractive spin systems, *Acta Appl. Math.* **2** (nos. 3/4) (1984), 379–390. Special issue edited by O. Brattelli and P. E. T. Jørgensen. The latter reference is a special issue on Positive Semigroups, and new results. It also contains an extended and updated bibliography on the subject.

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